

HOMOLOGICALLY OPTIMAL CATEGORIES OF SEQUENCES LEAD TO N -COMPLEXES

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ABSTRACT. We study the category of \mathbb{Z} -indexed sequences over an abelian category and certain generalized homology functors for this category of sequences which are indexed by positive integers a and b . By looking at the corresponding derived category, we show that there is an “optimal” subcategory of sequences for every choice of our generalized homology functors, namely, the category of N -complexes (sequences for which the differential d satisfies $d^N = 0$) where $N = a + b$. In this optimal case we show that our homology functors reduce to Kapranov’s homology functors $\ker d^a / \operatorname{im} d^b$.

1. INTRODUCTION

Homological algebra traditionally studies chain complexes over an abelian category \mathcal{A} , that is sequences of objects in \mathcal{A} equipped with a differential d satisfying $d^2 = 0$. Associated to this setup is the classical homology functor $\ker d / \operatorname{im} d$.

More generally one can study the so-called N -complexes for $N \in \{2, 3, 4, \dots\}$, that is sequences of objects in \mathcal{A} equipped with a differential d satisfying $d^N = 0$. In [Ka] Kapranov generalizes some of the tools of homological algebra for such N -complexes, inspired by their application in quantum groups. Since then, various aspects of the homological algebra of N -complexes were studied by various authors, see [DV1, DV2, DV3, DV4, CSW, Ti, Gi, HK] and references therein.

In this paper, for a fixed generalized homology functor of the form

$$(1) \quad H := \ker d^a / (\ker d^a \cap \operatorname{im} d^b) = (\ker d^a + \operatorname{im} d^b) / \operatorname{im} d^b$$

we ask the following question: Which “nice” subcategory \mathcal{B} of the category of sequences is the “minimal” one with the property that the corresponding derived category $\mathcal{D}_H(\mathcal{B})$ with respect to H inherits from \mathcal{A} the property of having enough projectives? One of our results, combined in Theorem 14 and Corollary 13, asserts that the category $\operatorname{Com}_N(\mathcal{A})$ of N -complexes of \mathcal{A} has these properties for $N = a + b$.

In fact, we even show more. Namely, in Theorem 7 and the subsequent proposition we prove that the category $\operatorname{Com}_N(\mathcal{A})$ is “optimal” in the sense that for any category \mathcal{B} containing $\operatorname{Com}_N(\mathcal{A})$ there is an equivalence

$$\mathcal{D}_H(\mathcal{B}) \simeq \mathcal{D}_H(\operatorname{Com}_N(\mathcal{A}))$$

of the corresponding derived categories with respect to our choice of H . In the category $\operatorname{Com}_N(\mathcal{A})$ we have $\operatorname{im} d^b \subset \ker d^a$ because $d^a d^b = d^N = 0$, consequently H in (1) reduces to Kapranov’s homology $\ker d^a / \operatorname{im} d^b$. In particular, if we choose $a = b = 1$ we end up with $\operatorname{Com}_2(\mathcal{A})$, the category of ordinary chain complexes together with classical notions of homology and derived category.

The paper is organized as follows: In Section 2 we collect basic definitions on N -complexes and their homology. The statement about derived equivalence is proved in Section 3. The statement on projective resolutions is proved in Section 4.

This paper is an adaptation of a part of author's Master Thesis [Mi].

2. PRELIMINARIES

2.1. Sequences and N -complexes. Given an abelian category \mathcal{A} , define the category $\text{Seq}(\mathcal{A})$ as follows:

An **object** C of $\text{Seq}(\mathcal{A})$, called a *sequence*, is a collection of objects $\{C_i\}_{i \in \mathbb{Z}}$ together with a collection of morphisms $\{d_i : C_i \rightarrow C_{i+1}\}_{i \in \mathbb{Z}}$ in \mathcal{A} (called the *differential* of C and denoted by d) which is usually depicted as follows:

$$\cdots \xrightarrow{d_{-2}} C_{-1} \xrightarrow{d_{-1}} C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots$$

For $C, D \in \text{Seq}(\mathcal{A})$, a **morphism** $f \in \text{Hom}_{\text{Seq}(\mathcal{A})}(C, D)$ is a commuting diagram over \mathcal{A} of the following form:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{-1} & \longrightarrow & C_0 & \longrightarrow & C_1 & \longrightarrow & \cdots \\ & & \downarrow f_{-1} & & \downarrow f_0 & & \downarrow f_1 & & \\ \cdots & \longrightarrow & D_{-1} & \longrightarrow & D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots \end{array}$$

The category $\text{Seq}(\mathcal{A})$ is the functor category from the path category of the graph

$$\cdots \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

to \mathcal{A} , in particular, it is abelian because \mathcal{A} is abelian.

For $N \in \{1, 2, 3, \dots\}$ we define the category $\text{Com}_N(\mathcal{A})$ of N -complexes over \mathcal{A} as the full subcategory of $\text{Seq}(\mathcal{A})$ which consists of all object which satisfy the condition $d_{i+N-1} \cdots d_{i+1} d_i = 0$ for all $i \in \mathbb{Z}$. As usual, we will just write $d^N = 0$ for the latter condition.

2.2. Homology for N -complexes. For 2-complexes, which are also usually called *chain complexes*, we have the inclusion

$$\text{im } d \hookrightarrow \ker d$$

and there is a functor $\text{Com}_2(\mathcal{A}) \rightarrow \text{Com}_1(\mathcal{A})$, called *homology*, given by

$$(2) \quad \ker d / \text{im } d.$$

For N -complexes, the differential induces the following poset of non-trivial inclusions:

(3)

Our goal is to study various generalizations of the classical homology given by (2) to N -complexes. To define such homologies we can use ingredients shown on the diagram (3) and also their various sums and intersections. For example, the principle of “throwing away as much information as possible” results in the homology

(4) $\ker d / (\ker d \cap \operatorname{im} d) .$

On the other hand, the principle of “keeping as much information as possible” results in the homology

(5) $\ker d^{N-1} / \operatorname{im} d^{N-1} .$

In some sense, the latter does preserve more information than, say

(6) $\ker d^{N-1} / \operatorname{im} d$

when $N > 2$. However, one should be aware of the fact that the homology (6) cannot be recovered only knowing the homology (5). To see this, consider the following pair of 3-complexes of abelian groups:

$$\begin{aligned} \cdots &\longrightarrow \mathbb{Z}/8\mathbb{Z} \xrightarrow{2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{2} \mathbb{Z}/8\mathbb{Z} \longrightarrow \cdots \\ \cdots &\longrightarrow 2\mathbb{Z}/4\mathbb{Z} \xrightarrow{0} 2\mathbb{Z}/4\mathbb{Z} \xrightarrow{0} 2\mathbb{Z}/4\mathbb{Z} \xrightarrow{0} 2\mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots \end{aligned}$$

It is easy to check that the homology $\ker d^2 / \operatorname{im} d^2$ from (5) has identical values for both of them, while the homology $\ker d^2 / \operatorname{im} d$ from (6) has different values on these two complexes.

Now fix $N \geq 1$. For positive integers a, b such that $a + b \geq N$, define the homology functor $H_j^{(a,b)} : \operatorname{Com}_N(\mathcal{A}) \rightarrow \mathcal{A}$, where $j \in \mathbb{Z}$, as follows:

(7) $H_j^{(a,b)} := \ker d^a / \operatorname{im} d^b \text{ (evaluated at position } j\text{)} .$

Proposition 1. *Equation (7) indeed defines a functor.*

Proof. Consider the functor

$$S^{(a,b)} : \operatorname{Com}_N(\mathcal{A}) \rightarrow \operatorname{Com}_2(\mathcal{A})$$

which maps an N -complex

(8) $C : \quad \cdots \xrightarrow{d} C_{-1} \xrightarrow{d} C_0 \xrightarrow{d} C_1 \xrightarrow{d} \cdots$

to the following 2-complex:

$$S^{(a,b)}C : \quad \cdots \xrightarrow{d^a} C_{-b} \xrightarrow{d^b} C_0 \xrightarrow{d^a} C_a \xrightarrow{d^b} \cdots$$

Further, $S^{(a,b)}$ maps a morphism

$$\begin{array}{ccccccc} C : & \cdots & \longrightarrow & C_{-1} & \longrightarrow & C_0 & \longrightarrow & C_1 & \longrightarrow & \cdots \\ f \downarrow & & & \downarrow f_{-1} & & \downarrow f_0 & & \downarrow f_1 & & \\ D : & \cdots & \longrightarrow & D_{-1} & \longrightarrow & D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots \end{array}$$

to the following morphism of 2-complexes:

$$\begin{array}{ccccccc} S^{(a,b)}C : & \cdots & \longrightarrow & C_{-b} & \longrightarrow & C_0 & \longrightarrow & C_a & \longrightarrow & \cdots \\ S^{(a,b)}f \downarrow & & & \downarrow f_{-b} & & \downarrow f_0 & & \downarrow f_a & & \\ S^{(a,b)}D : & \cdots & \longrightarrow & D_{-b} & \longrightarrow & D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots \end{array}$$

As the classical homology

$$H_0^{(1,1)} = \ker d / \operatorname{im} d : \operatorname{Com}_2(\mathcal{A}) \rightarrow \mathcal{A}$$

is functorial, we derive that

$$H_0^{(a,b)} = H_0^{(1,1)} S^{(a,b)}$$

is functorial as well.

Denote by T^j the (invertible) translation functor that translates a complex j steps “to the left”, that is position i moves to position $i - j$. With this notation we have

$$(9) \quad H_j^{(a,b)} = H_0^{(1,1)} S^{(a,b)} T^j.$$

The claim of the proposition follows. \square

The functors $H_j^{(a,b)}$ are called *Kapranov homology* functors.

Proposition 2. *The differential induces a natural transformation $H_j^{(a,b)} \rightarrow H_{j+1}^{(a,b)}$.*

Proof. Apply $H_j^{(a,b)}$ to the morphism $\underline{d}_C : C \rightarrow T^1 C$ given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{-1} & \xrightarrow{d} & C_0 & \xrightarrow{d} & C_1 & \longrightarrow & \cdots \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ \cdots & \longrightarrow & C_0 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_2 & \longrightarrow & \cdots \end{array}$$

Then, for each C , we have a sequence

$$\cdots \longrightarrow H_0^{(a,b)} T^{-1} C \xrightarrow{H_0^{(a,b)} \underline{d}_{T^{-1}C}} H_0^{(a,b)} C \xrightarrow{H_0^{(a,b)} \underline{d}_C} H_0^{(a,b)} T^1 C \longrightarrow \cdots$$

which is equal to the sequence

$$(10) \quad \cdots \longrightarrow H_{-1}^{(a,b)} C \xrightarrow{H_{-1}^{(a,b)} \underline{d}_C} H_0^{(a,b)} C \xrightarrow{H_0^{(a,b)} \underline{d}_C} H_1^{(a,b)} C \longrightarrow \cdots$$

\square

The sequence (10) is, in fact, an M -complex, where $M = \min(a, b)$. Hence $H_{-}^{(a,b)}$ defines a functor

$$H^{(a,b)} : \text{Com}_N(\mathcal{A}) \rightarrow \text{Com}_M(\mathcal{A}).$$

In the classical setting of chain complexes, that is in the category $\text{Com}_2(\mathcal{A})$, the functor $H^{(1,1)}$ maps 2-complexes to 1-complexes and, since $d = 0$ for 1-complexes, the above property is not really visible.

2.3. Total Kapranov homology. As was shown in [Ka], the Kapranov homology functors $H_j^{(a,b)}$ are connected by two families of commuting natural transformations. We recall this construction here.

Let C be an N -complex as in (8). The diagram (3) is a poset of inclusions. Therefore it commutes and we may consider this diagram as the following morphism of sequences

$$(11) \quad \begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \text{im } d^{N-1} & \longrightarrow & \text{im } d^{N-2} & \longrightarrow & \cdots & \longrightarrow & \text{im } d & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \ker d & \longrightarrow & \ker d^2 & \longrightarrow & \cdots & \longrightarrow & \ker d^{N-1} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

which we call α . Since $\text{Seq}(\mathcal{A})$ is abelian, α has a cokernel with objects being the homologies

$$\cdots \longrightarrow 0 \longrightarrow H_j^{(1,N-1)} \xrightarrow{i_*} H_j^{(1,N-2)} \xrightarrow{i_*} \cdots \xrightarrow{i_*} H_j^{(N-1,1)} \longrightarrow 0 \longrightarrow \cdots$$

which automatically defines the natural transformation i_* , the differential in the above cokernel sequence, between the corresponding homology functors.

The differential d adds a new dimension to diagram (11) by inducing the vertical morphisms in the following commutative diagram:

$$(12) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & \text{im } d^{N-1} & \longrightarrow & \text{im } d^{N-2} & \longrightarrow & \cdots & \longrightarrow & \text{im } d & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & & & \downarrow d & & \downarrow 0 \\ & & 0 & \longrightarrow & \text{im } d^{N-1} & \longrightarrow & \cdots & \longrightarrow & \text{im } d^2 & \longrightarrow & \text{im } d \longrightarrow 0 \end{array}$$

where the rows are coming from the top row of (11). The same thing can be done with the bottom row of (11):

$$(13) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & \ker d & \longrightarrow & \ker d^2 & \longrightarrow & \cdots & \longrightarrow & \ker d^{N-1} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & & & \downarrow d & & \downarrow 0 \\ & & 0 & \longrightarrow & \ker d & \longrightarrow & \cdots & \longrightarrow & \ker d^{N-2} & \longrightarrow & \ker d^{N-1} \longrightarrow 0 \end{array}$$

Indeed, diagrams (12) and (13) both commute and are sequence of sequences (we will call this bisequences in analogy with bicomplexes being complexes of complexes). Again, the morphism α in (11) defines a morphism of bisequences from the bisequence (12) to the bisequence (13). The cokernel of this morphism of bisequences results in the following bisequence with components being Kapranov

homology functors

(14)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_j^{(1,N-1)} & \xrightarrow{i_*} & H_j^{(2,N-2)} & \xrightarrow{i_*} & \dots \xrightarrow{i_*} H_j^{(N-1,1)} \longrightarrow 0 \\
 & & \downarrow d^* & & \downarrow d^* & & \downarrow d^* \\
 & & 0 & \longrightarrow & H_{j+1}^{(1,N-1)} & \xrightarrow{i_*} & \dots \xrightarrow{i_*} H_{j+1}^{(N-2,2)} \xrightarrow{i_*} H_{j+1}^{(N-1,1)} \longrightarrow 0
 \end{array}$$

where d^* denotes the maps induced by d .

Define a new sequence with the objects

$$\mathbf{H}_n = \bigoplus_{2j+p=n} H_j^{(p,N-p)}$$

at position n and the differential $\mathbf{H}_n \rightarrow \mathbf{H}_{n+1}$ given by adding up the i_* 's and the d^* 's between the corresponding summands. In [Ka] it is shown that this defines a functor

$$\mathbf{H} : \text{Com}_N(\mathcal{A}) \rightarrow \text{Com}_{N-1}(\mathcal{A}),$$

the *total Kapranov homology functor*.

2.4. Homology for Sequences. Let \mathcal{A} be an abelian category and N a positive integer. Let T^1 be the translation functor as defined above. Given an object C in $\text{Seq}(\mathcal{A})$, the differential of C gives rise to a morphism

$$\underline{d}_C : C \longrightarrow T^1 C$$

in $\text{Seq}(\mathcal{A})$ given by

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{d} & C_{-1} & \xrightarrow{d} & C_0 & \xrightarrow{d} & C_1 \xrightarrow{d} \dots \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 \dots & \xrightarrow{d} & C_0 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_2 \xrightarrow{d} \dots
 \end{array}$$

For simplicity we will omit indices and, for example, write \underline{d}^2 for $\underline{d}_{T^1 C} \underline{d}_C$.

Let \mathcal{B} be a full subcategory of $\text{Seq}(\mathcal{A})$ containing $\text{Com}_N(\mathcal{A})$. Denote by I the inclusion functor $\text{Com}_N(\mathcal{A}) \hookrightarrow \mathcal{B}$. For each object C in \mathcal{B} we have the diagram

$$(15) \quad \ker \underline{d}_C^N \xrightarrow{k_C} C \xrightarrow{\underline{d}^N} T^N C,$$

where k_C denotes the inclusion. Note that $\ker \underline{d}_C^N$ is an N -complex. As $\text{Com}_N(\mathcal{A}) \hookrightarrow \mathcal{B}$, it follows that the diagram (15) is, in fact, a diagram in \mathcal{B} .

Lemma 3. *Let $C, D \in \text{Seq}(\mathcal{A})$ and $f : C \rightarrow D$ be a morphism. Then there is a unique $f' : \ker \underline{d}_C^N \rightarrow \ker \underline{d}_D^N$ such that the following diagram commutes*

$$\begin{array}{ccccc}
 \ker \underline{d}_C^N & \xrightarrow{k_C} & C & \xrightarrow{\underline{d}^N} & T^N C \\
 \vdots \downarrow f' & & \downarrow f & & \downarrow T^N(f) \\
 \ker \underline{d}_D^N & \xrightarrow{k_D} & D & \xrightarrow{\underline{d}^N} & T^N D
 \end{array}$$

Proof. We have $\underline{d}^N f k_C = T^N(f) \underline{d}^N k_C = 0$ since $\underline{d}^N k_C = 0$. By the universal property of kernels, there is a unique f' such that $k_D f' = f k_C$. \square

Define

$$[\ker d^N] : \mathcal{B} \longrightarrow \text{Com}_N(A)$$

as follows:

- a sequence C is mapped to the N -complex $\ker \underline{d}_C^N$,
- a morphism $f : C \rightarrow D$ is mapped to f' given by Lemma 3.

Lemma 4. *We have that $[\ker d^N]$ is a functor.*

Proof. The composition $[\ker d^N](f)[\ker d^N](g)$ satisfies the universal property required for $[\ker d^N](fg)$ since the following diagram commutes:

$$\begin{array}{ccccc}
 \ker \underline{d}_B^N & \xrightarrow{k_B} & B & \xrightarrow{\underline{d}^N} & T^N B \\
 [\ker d^N](g) \downarrow & & \downarrow g & & \downarrow T^N(g) \\
 \ker \underline{d}_C^N & \xrightarrow{k_C} & C & \xrightarrow{\underline{d}^N} & T^N C \\
 [\ker d^N](f) \downarrow & & \downarrow f & & \downarrow T^N(f) \\
 \ker \underline{d}_D^N & \xrightarrow{k_D} & D & \xrightarrow{\underline{d}^N} & T^N D
 \end{array}$$

Because of the uniqueness, we deduce that $[\ker d^N](fg)$ and $[\ker d^N](f)[\ker d^N](g)$ coincide. Similarly one shows that $[\ker d^N]$ maps identities to identities. \square

3. DERIVED EQUIVALENCE

3.1. An auxiliary adjunction.

Proposition 5. *The pair $(I, [\ker d^N])$ is an adjoint pair of functors.*

Proof. Directly from the definition we see that $[\ker d^N]$ is the identity on $\text{Com}_N(\mathcal{A})$, hence

$$[\ker d^N]I = \text{Id}_{\text{Com}_N(\mathcal{A})}.$$

We claim that the unit of the adjunction is the identity natural transformation

$$id : \text{Id}_{\text{Com}_N(\mathcal{A})} \longrightarrow [\ker d^N]I = \text{Id}_{\text{Com}_N(\mathcal{A})},$$

and the counit of the adjunction is the inclusion

$$k : I[\ker d^N] \longrightarrow \text{Id}_{\mathcal{B}}$$

defined in Subsection 2.4.

Indeed, let $C, D \in \mathcal{B}$ and $f : C \rightarrow D$. Then $k_D[\ker d^N]I(f) = fk_C$ by construction. This means that k is indeed a natural transformation. Further, we have

$$k_I \circ I(id) = k_I = id_I$$

since the inclusion of a complex into itself is the identity. But also

$$[\ker d^N](k) \circ id_{[\ker d^N]} = [\ker d^N](k) = id_{[\ker d^N]}$$

because k is monic and $k \circ [\ker d^N](k) = k \circ id_{[\ker d^N]}$. \square

3.2. Localization of categories. Recall that CAT denotes the metacategory of all categories, see [ML]. Given a category \mathcal{C} and a subclass of morphisms $S \subset \text{Mor}(\mathcal{C})$, define the *localizing functor* $Q : \mathcal{C} \rightarrow \mathcal{C}[S]$ as the universal functor in CAT that maps morphisms $f \in S$ to isomorphisms, that is, $Q : \mathcal{C} \rightarrow \mathcal{C}[S]$ is a functor such that for any other functor $F : \mathcal{C} \rightarrow \mathcal{D}$ that maps morphisms in S to isomorphisms, there is a unique, up to isomorphism, functor $F' : \mathcal{C}[S] \rightarrow \mathcal{D}$ such that $F = F'Q$. The domain of Q , denoted $\mathcal{C}[S]$, is called the *localized category* for details.

It follows from the definition that if a localized category $\mathcal{C}[S]$ (with associated Q) exists, then it is unique up to isomorphism. The localized category can be constructed by adding formal inverses to morphisms in S , see [GM, III §2] for details.

3.3. Main result.

Theorem 6. *Let \mathcal{C} and \mathcal{D} be two categories and G and F a pair of functors as follows:*

$$\mathcal{C} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{F} \end{array} \mathcal{D}.$$

Assume we are given two natural transformations

$$\eta : \text{Id}_{\mathcal{D}} \rightarrow GF, \quad \varepsilon : FG \rightarrow \text{Id}_{\mathcal{C}}$$

and a pair of subclasses of morphisms

$$S \subset \text{Mor}(\mathcal{C}), \quad T \subset \text{Mor}(\mathcal{D})$$

such that

$$F(T) \subset S, \quad G(S) \subset T, \quad \{\varepsilon_X\}_{X \in \mathcal{C}} \subset S, \quad \{\eta_X\}_{X \in \mathcal{D}} \subset T.$$

Then the localized categories $\mathcal{C}[S]$ and $\mathcal{D}[T]$ are equivalent. Moreover, the same holds if we arbitrarily change the direction of η or ε .

Proof. Let $Q : \mathcal{C} \rightarrow \mathcal{C}[S]$ and $R : \mathcal{D} \rightarrow \mathcal{D}[T]$ be the corresponding localizing functors. Since RG maps morphisms from S to isomorphisms, there is a unique G' such that $RG = G'Q$. Similarly, there is a unique F' such that $QF = F'R$, that is the following diagrams commute

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ Q \downarrow & & \downarrow R \\ \mathcal{C}[S] & \xrightarrow{G'} & \mathcal{D}[T] \end{array} \quad \begin{array}{ccc} \mathcal{C} & \xleftarrow{F} & \mathcal{D} \\ Q \downarrow & & \downarrow R \\ \mathcal{C}[S] & \xleftarrow{F'} & \mathcal{D}[T] \end{array}$$

We have a natural transformation $Q(\varepsilon) : QFG \xrightarrow{=H} F'G'Q \rightarrow Q$. But Q acts as the identity on objects, so for each object X of $\mathcal{C}[S]$ we have an isomorphism

$$\varepsilon_X : F'G'X \rightarrow X$$

in $\mathcal{C}[S]$ because ε_X lies in S . To show that ε defines a natural transformation $F'G' \rightarrow \text{Id}_{\mathcal{C}[S]}$, it is enough to check that ε_X commutes with generators in $\mathcal{C}[S]$.

We already know that ε_X commutes with the morphisms of \mathcal{C} . Since morphisms from S are invertible in $\mathcal{C}[S]$, it follows that ε_X commutes with the inverses of all morphisms in S (indeed, $xs = ty$ implies $t^{-1}x = ys^{-1}$ if s and t are invertible). Since all ε_X are invertible in $\mathcal{C}[S]$ and ε is a natural transformation, similarly the inverses ε_X^{-1} constitute a natural transformation, call it ε^{-1} , and this natural transformation is the inverse of ε . Then

$$\varepsilon\varepsilon^{-1} = id_{\text{Id}_{\mathcal{C}}} \text{ and } \varepsilon^{-1}\varepsilon = id_{F'G'}$$

which proves that the functors $\text{Id}_{\mathcal{C}}$ and $F'G'$ are isomorphic. Similarly one shows that $\text{Id}_{\mathcal{D}}$ and $G'F'$ are isomorphic. This proves the first claim of the theorem.

Since the direction of ε (resp. η) did not matter in the argument above, the same argument can be applied if we arbitrarily change the direction of ε or η . \square

3.4. Derived categories. Let \mathcal{C} and \mathcal{D} be two categories and $H : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. A morphism f of \mathcal{C} is called an *H-quasi-isomorphism* if $H(f)$ is an isomorphism.

Further, we define the corresponding *derived category* $\mathcal{D}_H(\mathcal{C})$ as the category \mathcal{C} localized with respect to all H-quasi-isomorphisms, that is

$$(16) \quad \mathcal{D}_H(\mathcal{C}) := \mathcal{C}[S] \text{ where } S = \{f \in \text{Mor}(\mathcal{C}) : H(f) \text{ is an isomorphism}\}.$$

For example, for an abelian category \mathcal{A} the usual derived category $\mathcal{D}(\mathcal{A})$ is defined as $\mathcal{D}_{H(1,1)}(\text{Com}_2(\mathcal{A}))$, see [GM, III §2]. Similarly one can define derived categories of sequences or N -complexes choosing one's favorite homology functor.

3.5. Derived equivalence.

Theorem 7. *Let \mathcal{A} be an abelian category, $N \in \{1, 2, 3, \dots\}$ and \mathcal{B} be a full subcategory of $\text{Seq}(\mathcal{A})$ containing $\text{Com}_N(\mathcal{A})$. Then, given any category \mathcal{C} and any functor $E : \text{Com}_N(\mathcal{A}) \rightarrow \mathcal{C}$, there is an equivalence of categories*

$$\mathcal{D}_H(\mathcal{B}) \simeq \mathcal{D}_H(\text{Com}_N(\mathcal{A}))$$

where $H = E[\ker d^N]$.

Proof. Recall that $[\ker d^N]$ acts as the identity functor on $\text{Com}_N(\mathcal{A})$. We want to apply Theorem 6. The adjoint pair $(I, [\ker d^N])$

$$\mathcal{B} \begin{array}{c} \xrightarrow{[\ker d^N]} \\ \xleftarrow{I} \end{array} \text{Com}_N(\mathcal{A}),$$

given by Proposition 5, and the corresponding adjunction morphisms provide the functors and, respectively, the natural transformations as mentioned in the formulation of Theorem 6. The subclasses of morphisms to be localized are the H-quasi-isomorphisms as described in (16). The functor $[\ker d^N] : \mathcal{B} \rightarrow \text{Com}_N(\mathcal{A})$ maps quasi-isomorphisms to quasi-isomorphisms because $H[\ker d^N] = H$. The functor $I : \text{Com}_N(\mathcal{A}) \rightarrow \mathcal{B}$ maps quasi-isomorphisms to quasi-isomorphisms because $HI = E[\ker d^N]I = E = H$.

Finally, let us check that the natural transformations id and k consist of quasi-isomorphisms. For id it is obvious and for k it holds because $[\ker d^N](k) = id$. Therefore we may apply Theorem 6 which yields the desired statement. \square

The following proposition deals with the case $E = H^{(a,b)}$ in Theorem 7 which is of particular interest.

Proposition 8. *Let $N = a + b$. Then the functor $H^{(a,b)}[\ker d^N]$ can be written in the form*

$$\ker d^a / (\ker d^a \cap \text{im } d^b) \text{ or } (\ker d^a + \text{im } d^b) / \text{im } d^b.$$

Proof. The two expressions are equal by the second isomorphism theorem. We show this for the left hand side expression. Let C be a sequence and denote the

differential of C and $[\ker d^N]C$ by d and δ , respectively. Then $\ker d^a = \ker \delta^a$ because $\ker d^a \subset \ker d^N$. In general $\operatorname{im} d^b \neq \operatorname{im} \delta^b$ but we have

$$\ker d^a \cap \operatorname{im} d^b = \ker \delta^a \cap \operatorname{im} \delta^b = \operatorname{im} \delta^b$$

because $a + b = N$. \square

Corollary 9. *Let homology on $\operatorname{Com}_N(\mathcal{A})$ be given by $H = \ker d / (\ker d \cap \operatorname{im} d)$. Then there is an equivalence of categories*

$$\mathcal{D}_H(\operatorname{Com}_N(\mathcal{A})) \simeq \mathcal{D}_H(\operatorname{Com}_2(\mathcal{A})).$$

Proof. By Proposition 8 we have $H = H^{(1,1)}[\ker d^2]$. The statement follows from Theorem 7 with $\mathcal{B} = \operatorname{Com}_N(\mathcal{A})$. \square

Corollary 10. *There is an equivalence of categories*

$$\mathcal{D}_{[\ker d^N]}(\mathcal{B}) \simeq \operatorname{Com}_N(\mathcal{A}).$$

Proof. This follows from Theorem 7 by taking $\mathcal{C} = \operatorname{Com}_N(\mathcal{A})$ and $E = \operatorname{Id}_{\operatorname{Com}_N(\mathcal{A})}$ as under this choice we have $\mathcal{D}_H(\operatorname{Com}_N(\mathcal{A})) = \operatorname{Com}_N(\mathcal{A})$ since $\operatorname{Com}_N(\mathcal{A})$ has the universal property required for the localized category. \square

4. PROJECTIVE RESOLUTIONS

We continue to work with categories \mathcal{A} , $\operatorname{Seq}(\mathcal{A})$ and $\operatorname{Com}_N(\mathcal{A})$ as described above. Denote by $R_N : \operatorname{Com}_2(\mathcal{A}) \rightarrow \operatorname{Com}_N(\mathcal{A})$ the functor mapping a chain complex

$$C : \quad \cdots \xrightarrow{d} C_{-1} \xrightarrow{d} C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots$$

to the N -complex $R_N C$ where

$$(R_N C)_{Nj} := C_{2j} \quad \text{for } j \in \mathbb{Z},$$

and the odd degrees of C are repeated $N - 1$ times as follows:

$$\begin{array}{ccccccccccc} R_N C : & \cdots & \xrightarrow{1} & C_{-1} & \xrightarrow{d} & C_0 & \xrightarrow{d} & C_1 & \xrightarrow{1} & \cdots & \xrightarrow{1} & C_1 & \xrightarrow{d} & C_2 & \xrightarrow{d} & \cdots \\ \text{degree} & & & -1 & & 0 & & 1 & & & & N-1 & & N & & \end{array}$$

The action of R_N on morphisms is defined accordingly.

Proposition 11. *For $a, b \in \{1, 2, 3, \dots\}$ the functor $R_{(a+b)}$ maps ordinary quasi isomorphisms of chain complexes to $H^{(a,b)}$ -quasi isomorphisms of $(a+b)$ -complexes.*

Proof. This follows from the definitions and (9) by a direct computation. \square

Recall that an abelian category \mathcal{A} is said to have *enough projectives* provided that each object in \mathcal{A} is a quotient of a projective object. A natural analogue of this standrd notions for sequences over \mathcal{A} looks as follows:

Definition 12. Let \mathcal{B} be a full subcategory of $\operatorname{Seq}(\mathcal{A})$, \mathcal{C} any category and $H : \mathcal{B} \rightarrow \mathcal{C}$ a functor. An object $P \in \mathcal{B}$ is said to be a *H-projective resolution* of the object $X \in \mathcal{A}$ if P_i is equal to zero for $i \geq 1$, P_i is projective for $i \leq 0$ and there is an H-quasi isomorphisms in \mathcal{B} as follows

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_{-2} & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Corollary 13. *Let $a, b \in \{1, 2, 3, \dots\}$. If \mathcal{A} has enough projectives, then every object in \mathcal{A} , considered as an object in $\text{Com}_{a+b}(\mathcal{A})$ in the obvious way, admits an $H^{(a,b)}$ -projective resolution.*

Proof. If some object $P \in \text{Com}_2(\mathcal{A})$ is an $H^{(1,1)}$ -projective resolution of $X \in \mathcal{A}$ (i.e. P is a projective resolution of X in the usual sense), then, by Proposition 11, the object $R_{(a+b)}P \in \text{Com}_{a+b}(\mathcal{A})$ is an $H^{(a,b)}$ -projective resolution of X . \square

The following statement gives a kind of a “lower bound on N ” for existence of projective resolutions.

Theorem 14. *Let $X \in \mathcal{A}$ be a non-projective object and $P \in \text{Com}_{a+b}(\mathcal{A})$ an $H^{(a,b)}$ -projective resolution of X . Then $P \notin \text{Com}_{a+b-1}(\mathcal{A})$.*

Proof. We need to find a position at which the differential d of P satisfies

$$d^{a+b-1} \neq 0.$$

For the case $a = 1$ we have

$$H_0^{(a,b)}(P) = P_0/d^b P_{-b} \simeq X$$

but $d^b P_{-b} = d^{a+b-1} P_{-b} \neq 0$ since X was not projective.

Now assume $a \geq 2$. Homology of P at position $1 - a$ is

$$H_{1-a}^{(a,b)}(P) = P_{1-a}/d^b P_{1-a-b} = 0$$

and hence

$$d^b : P_{1-a-b} \rightarrow P_{1-a}$$

is epi. If $1 - a < -b$, homology of P at position $1 - a + b$ is again zero and hence

$$d^b : P_{1-a} \rightarrow P_{1-a+b}$$

is epi. We may now repeat this n more steps until $1 - a + nb \geq -b$. But, similarly to the case of $a = 1$, $d^b : P_{-b} \rightarrow P_0$ is non-zero. This means, in particular, that $d^{-1+a-nb} : P_{1-a+nb} \rightarrow P_0$ is non-zero and

$$d^{-1+a-nb}(d^b)^{n+1} = d^{a+b-1} : P_{a+b-1} \rightarrow P_0$$

is non-zero. The claim follows. \square

The combination of Theorems 7 and 14 suggests that the category $\text{Com}_N(\mathcal{A})$ is “homologically optimal” with respect to (generalized) Kapranov homology functors in the sense that, on the one hand, it is big enough so that the derived category inherits the property of having projective resolutions and, on the other hand, it is small enough in the sense the derived category of any bigger category produces an equivalent category.

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